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Solving the functional Schrödinger equation: Yang-Mills string tension and surface critical scaling

Paul Mansfield

Centre for Particle Theory, University of Durham
Durham DH1 3LE, U.K.
E-mail: P.R.W.Mansfield@durham.ac.uk

Abstract: Motivated by a heuristic model of the Yang-Mills vacuum that accurately describes the string-tension in three dimensions we develop a systematic method for solving the functional Schrödinger equation in a derivative expansion. This is applied to the Landau-Ginzburg theory that describes surface critical scaling in the Ising model. A Renormalisation Group analysis of the solution yields the value $\eta = 1.003$ for the anomalous dimension of the correlation function of surface spins which compares well with the exact result of unity implied by Onsager's solution. We give the expansion of the corresponding β-function to 17-th order (which receives contributions from up to 17-loops in conventional perturbation theory).

Keywords: Renormalization Group, Field Theories in Lower Dimensions, Confinement, Boundary Quantum Field Theory.

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1. Introduction

In recent years there have been considerable advances in our understanding of the quantum field theory of systems with higher conserved charges. Deep mathematical structures have emerged in the study of integrable models in two dimensions and in supersymmetric theories in higher dimensions. However the field theories that are directly applicable to the physical world do not manifest the high degree of symmetry needed for mathematical tractability. This limits us to the use of perturbation theory, which cannot describe strongly interacting theories, and the largely numerical approach of lattice field theory. This suggests that it is worthwhile to develop new methods for the solution of generic non-integrable quantum field theories. In this paper we will describe an approach to quantum field theory in which the functional Schrödinger equation is constructed in an essentially standard way, [1]–[16], but is then solved using a derivative expansion for the wave-functional from which the continuum limit is obtained using a version of the Borel transform. Like perturbation theory this yields an approximation in the form of a series, but unlike perturbation theory the method applies at arbitrary values of the coupling. The approach is inspired by a heuristic model of the Yang-Mills vacuum but we will develop it for the toy-model of scalar φ^4 theory in $1 + 1$ dimensions. This is partly because the simplicity of the model allows the series expansion to be calculated to reasonably high order using modest computing resources but also because the strong-coupling behaviour of the vacuum wave-functional describes the surface critical scaling of the Ising model and so we are able to test the method against exact results which derive from the Onsager solution and conformal field theory.

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In a D-dimensional gauge theory there is a constant force between static quarks leading to confinement when the Wilson loop, $WL[C]$, associated with a closed curve, C, depends on the minimal area spanning C as $\exp(-\sigma \mathrm{Area}[C])$, [17]. σ is the string tension. $WL[C]$ is the vacuum expectation value of the trace of the path-ordered exponential of the gauge-field, $WL[C] = \langle 0|\,\mathrm{tr}\, P\exp(-\int_C A\cdot dx)\,|0\rangle$. This has a functional integral representation $\int \mathcal{D}A\exp(-S_D/g^2)\,\mathrm{tr}\,P\exp(-\int_C A\cdot dx)$ where S_D is the Yang-Mills action in D euclidean space-time dimensions, and g a coupling. Areal behaviour is observed in Monte Carlo lattice studies for $D=2$, 3 and 4, but has not been obtained by analytic calculation except in two dimensions where Yang-Mills theory becomes free with the gauge choice $A_1 = 0$. In 1979 Greensite [18] and Halpern [19] proposed a simple model of the confining Yang-Mills vacuum. For simplicity consider $D=3$. If we choose the temporal gauge $A_0 = 0$ and work in the Schrödinger representation so that A_1 and A_2 are diagonal on the quantisation surface $t=0$ then the representative of the vacuum $\langle A|0\rangle = \Psi[A]$ is a gauge-invariant functional of $A_1(x^1,x^2)$ and $A_2(x^1,x^2)$. The idea is to model this three-dimensional vacuum in terms of the two-dimensional action as $\Psi \approx \exp(-kS_2/g^4)$, with k a dimensionless constant. This guarantees gauge-invariance and if we choose a planar C so that it can be rotated into the quantisation surface then the usual expression of vacuum expectation values within the Schrödinger representation yields

$$\begin{aligned} WL[C] &= \int \mathcal{D}A_1\,\mathcal{D}A_2\,\Psi^*[A]\,\mathrm{tr}\,P\exp\left(-\int_C A\cdot dx\right)\Psi[A] \\ &= \int \mathcal{D}A_1\,\mathcal{D}A_2\exp\left(-\frac{2kS_2}{g^4}\right)\mathrm{tr}\,P\exp\left(-\int_C A\cdot dx\right) \end{aligned} \tag{1.1}$$

which reduces to the Wilson-loop for a two-dimensional gauge theory for which we can compute the string tension in terms of k. This form of $\Psi[A]$ was compared directly to Monte Carlo simulations of the vacuum functional in lattice gauge theory in [20]. It might now be timely to repeat this comparison given the improvements in lattice gauge theory technology. This argument was generalised to quantum gravity in [21]–[26].

$\Psi[A]$ should satisfy the Schrödinger equation, and this has been used to estimate k, [27]. The hamiltonian is $-\int d^2x\,\mathrm{tr}(\dot{A}^2 + B^2)$ where the 'magnetic field', B, is the $1,2$-component of the field strength from which the two-dimensional action is constructed as $S_2 = -\int d^2x\,\mathrm{tr}(B^2)$. In the Schrödinger representation $\dot{A} = -ig^2\delta/\delta A$, so that the Schrödinger equation is

$$H\,\Psi = \int d^2x\,\mathrm{tr}\left(g^2\frac{\delta^2}{\delta A^2} - \frac{1}{g^2}B^2\right)\Psi = E\,\Psi\,. \tag{1.2}$$

Differentiating S_2 functionally with respect to A gives the covariant derivative of B, D_iB. Differentiating again gives a second order partial differential operator, Ω, so that for the model vacuum the Schrödinger equation becomes

$$H\,\Psi = \left(\frac{k}{g^2}\,\mathrm{Tr}\,\Omega + \frac{1}{g^6}k^2\int d^2x\,\mathrm{tr}((DB)^2) + \frac{1}{g^2}S_2\right)\Psi = E\,\Psi\,. \tag{1.3}$$

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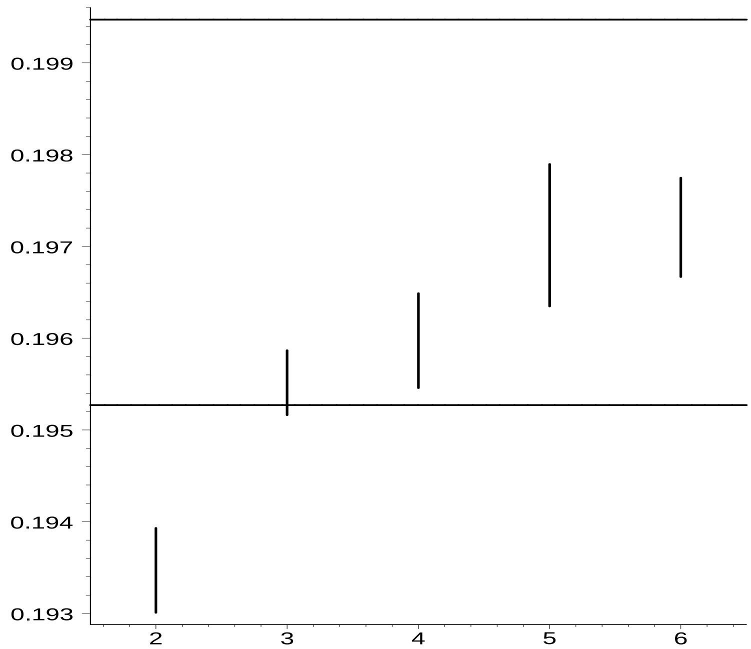

Figure 1: $\sqrt{\sigma}/(g^2\sqrt{N^2-1})$ against N from lattice QCD, [28]. The horizontal lines correspond to the predictions from σ_ζ (lower) and σ_{KKN}.

When the functional trace of Ω is regulated using a heat-kernel based regulator to preserve gauge invariance the result is a divergent constant plus a multiple of S_2, so that k, and hence σ, are determined if we ignore $(DB)^2$ as a higher order correction. Specifically, if zeta-function regularisation [27] is used we obtain

$$\sigma = \frac{23g^4(N^2-1)}{192\pi} \equiv \sigma_\zeta \tag{1.4}$$

for the gauge-group $\mathrm{SU}(N)$, (and where we have adapted the result of [27] to the convention $\mathrm{tr}(T_a T_b) = -\delta_{ab}/2$ for the generators of the fundamental representation.) This formula shows remarkable agreement with recent lattice studies. Figure 1 plots $\sqrt{\sigma}/(g^2 N)$ against N as measured in Monte Carlo simulations [28]. $\sqrt{\sigma_\zeta}/(g^2N)$ is shown on the same figure, and the values agree to within a half to one percent which is significant given that the formula (1.4) has no adjustable parameters. (There is a similar formula, $\sigma_{KKN} = g^4(N^2-1)/(8\pi)$, [29],based on a different parametrisation of the canonical variables which is also displayed in the figure).

This model begs many questions: Can the vacuum functional really be the exponential of the integral of a local quantity, i.e. depending on a single point in space, since this is quite unlike the non-local functionals seen in perturbation theory. For example the free Maxwell field in four-dimensions has

$$\Psi = \exp\left(-\int d^3x\, d^3y \frac{B(x)\,B(y)}{(e^2|x-y|^2)}\right).$$

Is the approach regularisation dependent, and is it consistent with the usual ultra-violet behaviour and renormalisation of gauge theories? Is the model Lorentz invariant? Can

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corrections to the model be computed and shown to converge to the lowest energy solution to the Schrödinger equation? Applying the argument to four-dimensional gauge theory would give a vacuum functional that is the exponential of the three-dimensional action with a cut-off dependent coefficient that appears to lead to a divergent string tension. Despite these issues the agreement between lattice simulations and the formula for the string tension suggests that it is worthwhile to investigate the approach further and make it more systematic.

In calculating k the contribution of $(DB)^2/g^6$ was ignored in (1.3). This may be corrected for by including a similar term in the vacuum functional. This leads to an adjustment to the coefficient of the original term proportional to $\mathrm{tr}(B^2)$ because $\int d^2x\, \delta^2/\delta A^2$ applied to $\int d^2x\, \mathrm{tr}((DB)^2)$ gives a piece proportional to $\epsilon^{-2}\int d^2x\, \mathrm{tr}\, B^2$, where ϵ is a short-distance cut-off as well as $\int d^2x\, \mathrm{tr}((DB)^2)$ so that now, if we set $\Psi = \exp W$ then the approximation for W is

$$W \approx \int d^2x\, \mathrm{tr}\left(\left(\frac{k}{g^4} + \frac{k_3}{\epsilon^2 g^8}\right) B^2 + \frac{k_2}{g^8}(DB)^2\right),$$

which satisfies the Schrödinger equation for appropriate values of k_2 and k_3 if now we ignore terms of mass dimension eight. These higher dimension expressions can then be taken into account by adding further terms so that the process generates an expansion for W consisting of local expressions of increasing dimension with coefficients that are themselves series in decreasing powers of the cut-off ϵ. Superficially such a derivative expansion would appear to be applicable only to slowly varying fields, making it difficult to address the ultra-violet behaviour that has to be understood in order to renormalise the theory. Furthermore the cut-off dependence of the coefficients is unexpected since a theorem due to Symanzik [30] implies that W should be finite as $\epsilon \to 0$ (at least to all orders in perturbation theory).

To address the problem of whether a systematic expansion of the vacuum functional can be obtained as a derivative expansion with the correct ultra-violet behaviour we turn to a simpler model than gauge theory, namely scalar φ^4 theory in two space-time dimensions. The simplicity of the model will allow us to pursue the expansion to high order and, as stated earlier, the strong-coupling ultra-violet behaviour is related to the surface critical scaling of the Ising model, providing exact results to test our approach against.

The hamiltonian for scalar φ^4 in two space-time dimensions is

$$H = \int dx \left(\frac{1}{2}\left(\dot\varphi_0^2 + \varphi_0'^2 + m_0^2\,\varphi_0^2\right) + \frac{g_0}{4!}\,\varphi_0^4\right), \tag{1.5}$$

where φ_0, m_0 and g_0 are the bare field, mass and coupling. Quantisation consists simply of replacing $\dot\varphi_0$ by $-i\delta/\delta\varphi_0$. We can now look for eigenfunctionals of H. We will assume that, as in quantum mechanics, the vacuum functional has no nodes, so that it can be written as $\Psi = \exp W$, giving a Schrödinger equation

$$\int dx\left(-\frac{\delta^2 W}{\delta\varphi_0^2} - \left(\frac{\delta W}{\delta\varphi_0}\right)^2 + \varphi_0'^2 + m_0^2\,\varphi_0^2 + \frac{g_0}{12}\,\varphi_0^4\right) = E_v \tag{1.6}$$

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where E_v is the energy of the vacuum. The first term needs to be regularised, and we adopt a simple momentum cut-off, replacing $\delta^2/\delta\varphi_0^2$ by

$$\left.\frac{\delta^2}{\delta\varphi_0^2}\right|_s \equiv \int dx\, dy \int_{|k|<1/s} \frac{dk}{2\pi} e^{ik(x-y)} \frac{\delta^2}{\delta\varphi_0(x)\,\delta\varphi_0(y)} \tag{1.7}$$

with s a small distance.

The problem of renormalisation is to tune the bare quantities in terms of s to obtain finite physics. Renormalisation conditions have also to be chosen to replace the (potentially divergent) bare quantities by finite parameters, in this case a mass scale μ and dimensionless coupling g. We are aided in this by Symanzik's theorem [30] which implies that provided we renormalise φ_0 multiplicatively $\varphi_0 = \sqrt{Z}\,\varphi$ (and tune m_0 and g_0) then W is finite apart from a local term proportional to $\int dx\, \varphi^2$.

Conventional semi-classical perturbation theory can be developed [1] by re-instating $\hbar$ in the quantisation $\dot{\varphi}_0 = -i\hbar\delta/\delta\varphi_0$ and expanding $W = W_1/\hbar + W_2 + \hbar W_3 + ..$ so that to leading order (1.6) becomes the Hamilton-Jacobi equation with a solution most conveniently expressed in terms of the Fourier transform $\tilde{\varphi}$ as

$$W_1 = -\int \frac{dp}{4\pi}\tilde{\varphi}(-p)\omega(p)\tilde{\varphi}(p) - \frac{g_0}{4!}\int \frac{dp_1}{2\pi}\cdots\frac{dp_4}{2\pi} 2\pi\delta(p_1+\cdots+p_4)\,\frac{\tilde{\varphi}(p_1)\cdots\tilde{\varphi}(p_4)}{\omega(p_1)+\cdots+\omega(p_4)} + \cdots \tag{1.8}$$

where $\omega = \sqrt{p^2+m_0^2}$, and bare and renormalised quantities coincide at this (tree) level. Clearly for slowly varying φ we can take $|p| \ll m_0$ and expand $\omega = m_0 + p^2/(2m_0) + ..$ to obtain a local expansion. The one-loop, i.e. order $\hbar$, correction can be readily obtained, [1, 36], but this is not our purpose here. Instead we will develop the approach of [32] and consider solving for the full W starting from an expansion in a basis of local functionals of the renormalised field with coefficients $\{b\}$:

$$W = \int dx\, \left(b_2\varphi^2 + b_4\varphi^4 + b_6\varphi^6 + \cdots + b_{2,2}\varphi'^2 + b_{4,2}\varphi'^2\varphi^2 + \cdots + b_{2,4}\varphi''^2 + \cdots\right). \tag{1.9}$$

Now the regulated second functional derivative acts on this local expansion to produce another local expansion, for example

$$\int dx \left.\frac{\delta^2}{\delta\varphi^2}\right|_s \int dy\, \varphi^2\,\varphi'^2 = \int dx\, \left(\frac{2}{\pi s}\,\varphi'^2 + \frac{2}{3\pi s^3}\,\varphi^2\right). \tag{1.10}$$

The other terms in the Schrödinger equation are also local functionals of φ so we can reduce this functional differential equation to an infinite set of algebraic equations for the infinite number of constants $\{b\}$ by equating coefficients of the basis functionals to zero. To solve these requires some form of truncation, for example by simply ignoring coefficients of local terms containing more than a prescribed number of fields or derivatives. In [32] a different truncation was proposed that would use an estimate of such 'high order' coefficients. Once the problem has been made finite the equations can be solved, for example numerically, although the stability and convergence of the solutions may be sensitive to the truncation. We will use a different approach to that of [32]. Rather than truncate the expansion (1.9) we look for solutions to the full set of (untruncated) equations in the form of expansions

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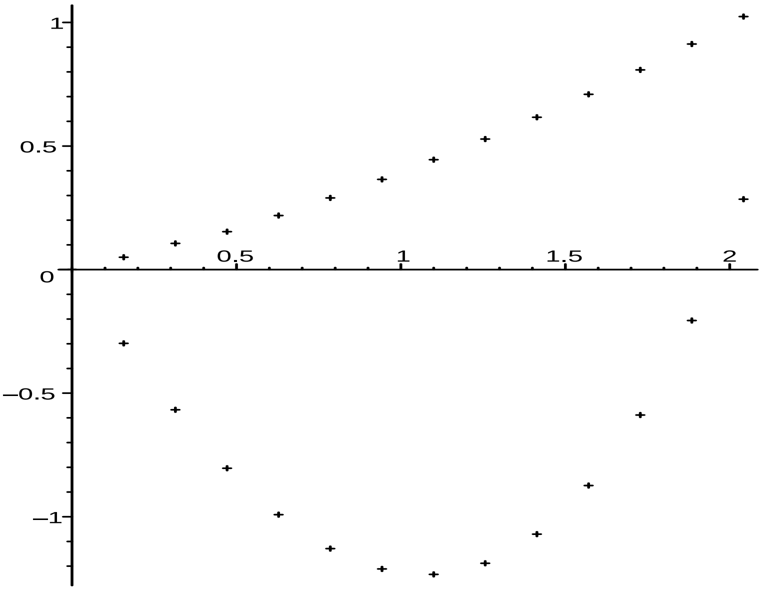

Figure 2: $\gamma(g)$ (upper curve) and $\beta(g)$ from the Schrödinger equation.

of the coefficients $\{b\}$ as power series in $1/s$. We will see that the enlarged set of equations obtained by equating to zero the various powers of s form closed systems that can be solved rapidly and simply in terms of the renormalised quantities μ and g. The quantities Z, g_0 and m_0 will similarly be obtained as power series in $1/s$. We will compute these series up to some fixed order, N, determined by computing resources. This is the only truncation we will use.

The continuum limit is obtained by sending s to zero, (and N to infinity) so these series solutions would appear to have little physical relevance, as they are only likely to converge for large s. However the small s behaviour can be extracted using a version of the Borel transform, which we describe in section 3.

We will test our solution of the Schrödinger equation against exact results obtained for the surface critical scaling of the Ising model from the Onsager solution and conformal field theory [31]. At a critical value of the renormalised coupling, g, the $\mu \to 0$ limit of φ^4 theory describes the Ising model at criticality. More specifically the wave-functional describes the Ising model with a boundary (corresponding to the quantisation surface) at the 'Ordinary Transition'. That part of the wave-functional that is quadratic in φ is related to the correlator of two spins lying in the boundary which is known to depend on their separation $|x|$ as $1/|x|$. This quadratic piece can be written as $\int dx\, dy\, \varphi(x)\varphi(y)/|x-y|^{2-\eta}$ where η is the anomalous dimension, so the correspondence with the Ising model requires that $\eta = 1$. Now a standard Renormalisation Group argument presented in section 4 gives η as the value of $d\log Z/d\log\mu \equiv \gamma(g)$ computed at the zero of the beta-function $dg/d\log\mu \equiv \beta(g)$. In figure 2 we display these two functions computed from our solution when we truncate the series expansions to order $1/s^{17}$ terms. The graph indicates good agreement with the exact result, and as we will see the agreement is considerably improved upon extrapolation in the number of terms, N.

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2. Solving the Schrödinger equation for φ^4.

The Schrödinger equation for the vacuum, $H \exp W = E_v \exp W$, can be written in terms of the renormalised field as

$$\int dx \left(\left. \frac{\delta^2 W}{\delta\varphi^2} \right|_s + \left(\frac{\delta W}{\delta\varphi} \right)^2 - Z^2 \varphi'^2 - m_0^2 Z^2 \varphi^2 - \frac{g_0}{12} Z^3 \varphi^4 + 2Z\mathcal{E}_v \right) = 0 \tag{2.1}$$

where $\mathcal{E}_v$ is the energy density of the vacuum. The local expansion (1.9) takes the form

$$\begin{aligned} W = \int dx \Big(& b_2\,\varphi^2 + b_4\,\varphi^4 + b_6\,\varphi^6 + \cdots + b_{2,2}\,\varphi'^2 + b_{2,4}\,\varphi''^2 + \cdots + \\ & + b_{4,2}\,\varphi'^2\,\varphi^2 + b_{4,4,1}\,\varphi''^2\,\varphi^2 + b_{4,4,2}\,\varphi'^4 + \cdots + \\ & + b_{6,2}\,\varphi'^2\,\varphi^4 + \cdots + b_8\,\varphi^8 + b_{8,2}\,\varphi'^2\,\varphi^6 + \cdots \Big) \end{aligned} \tag{2.2}$$

where the dots denote an infinite number of terms containing increasing powers of the field and its derivatives. The two (or three) labels on the coefficients denote the number of fields, the number of derivatives (and a further label to distinguish between different terms for which the first two labels are the same). We will not truncate the expansion at this stage. Substituting (1.9) into the Schrödinger equation gives

$$\begin{aligned} \int dx \Bigg(\Bigg[& \left(\frac{2b_2}{\pi s} + \frac{2b_{2,2}}{3\pi s^3} + \cdots \right) + \varphi^2 \left(\frac{12b_4}{\pi s} + \frac{2b_{4,2}}{3\pi s^3} + \cdots \right) + \\ & + \varphi'^2 \left(\frac{2b_{4,2}}{\pi s} + \frac{4b_{4,4,1}}{\pi s^3} + \frac{8b_{4,4,2}}{3\pi s^3} + \cdots \right) + \varphi^4 \left(\frac{30b_6}{\pi s} + \frac{2b_{6,2}}{3\pi s^3} + \cdots \right) + \\ & + \varphi^6 \left(\frac{56b_8}{\pi s} + \frac{2b_{8,2}}{3\pi s^3} + \cdots \right) + \varphi^2\varphi'^2 \left(\frac{12b_{6,2}}{\pi s} + \cdots \right) + \cdots \Bigg] + \\ & + \Big[\varphi^2 4b_2^2 + \varphi'^2 8b_2 b_{2,2} + \varphi^4 16 b_2 b_4 + \varphi^6 \left(24 b_2 b_6 + 16 b_4^2 \right) + \\ & \quad + \varphi^2\varphi'^2 \left(16 b_2 b_{4,2} + 48 b_{2,2} b_4 \right) + \cdots \Big] - \\ & - Z^2\varphi'^2 - m_0^2 Z^2 \varphi^2 - \frac{g_0}{12} Z^3 \varphi^4 + 2Z\mathcal{E}_v \Bigg) = 0\,. \end{aligned} \tag{2.3}$$

We can also look for an eigenstate of the hamiltonian describing a single particle at rest in the form $\Psi = U \exp W$ The Schrödinger equation for this state, $H\Psi = (\mu + E_v)\Psi$, can be written as

$$\int dx \left(\left. \frac{\delta^2 U}{\delta\varphi^2} \right|_s + 2\frac{\delta W}{\delta\varphi}\frac{\delta U}{\delta\varphi} + 2\mu Z U \right) = 0\,. \tag{2.4}$$

Take U as another local expansion:

$$U = \int dx \Big(\varphi + c_3\varphi^3 + c_5\varphi^5 + \cdots + c_{3,2}\varphi\varphi'^2 + c_{3,4}\varphi\varphi''^2 + \cdots + c_{5,2}\varphi^3\varphi'^2 + \cdots \Big)\,, \tag{2.5}$$

where again the dots denote an infinite number of local terms all of which we keep in the expansion for the time being. Substituting into the Schrödinger equation gives

$$\int dx \Bigg(\Bigg[\varphi \left(\frac{6c_3}{\pi s} + \frac{2c_{3,2}}{3\pi s^3} + \frac{2c_{3,4}}{5\pi s^5} + \cdots \right) + \varphi^3 \left(\frac{20c_5}{\pi s} + \frac{2c_{5,2}}{3\pi s^3} + \cdots \right) +$$

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$$+\varphi\varphi'^2\left(\frac{6c_{5,2}}{\pi s}+\cdots\right)+\cdots\Big]+$$
$$+2\Big[\varphi 2b_2+\varphi^3\left(b_4+6c_3b_2\right)+\varphi\varphi'^2\left(12c_3b_{2,2}+6c_{3,2}b_2+2b_{4,2}\right)+\cdots\Big]+$$
$$+2\mu Z\left(\varphi+c_3\varphi^3+c_5\varphi^5+\cdots+c_{3,2}\varphi\varphi'^2+c_{3,4}\varphi\varphi''^2+\cdots\right)\Big)=0\,. \tag{2.6}$$

These equations have to be solved in conjunction with renormalisation conditions that identify parameters that remain finite as the cutoff is removed. We will take μ to be our finite mass-scale. Symanzik's theorem [30] implies that the coefficients b in the local expansion of W are finite, with the possible exception of b_2, so we choose renormalisation conditions

$$b_4=-\frac{g\mu}{8}\,,\qquad b_{2,2}=-\frac{1}{4\mu}\,, \tag{2.7}$$

with dimensionless g playing the rôle of renormalised coupling. Our aim is to solve (2.3) and (2.6) for the coefficients b and c, and also for g_0, m_0 and Z in terms of the finite parameters μ and g.

Equating the coefficients of the basis functionals to zero in (2.3) gives an infinite set of algebraic equations. This may be divided into two classes. In the first class are the equations obtained from the coefficients of 1, φ^2, φ'^2, and φ^4.

$$\begin{aligned}\frac{2b_2}{\pi s}+\frac{2b_{2,2}}{3\pi s^3}+\cdots+2Z\mathcal{E}_v&=0\,,\\ \frac{12b_4}{\pi s}+\frac{2b_{4,2}}{3\pi s^3}+\cdots+4b_2^2-m_0^2Z^2&=0\,;\end{aligned} \tag{2.8}$$

$$\begin{aligned}\frac{2b_{4,2}}{\pi s}+\frac{4b_{4,4,1}}{\pi s^3}+\frac{8b_{4,4,2}}{3\pi s^3}+\cdots+8b_2b_{2,2}-Z^2&=0\,,\\ \frac{30b_6}{\pi s}+\frac{2b_{6,2}}{3\pi s^3}+\cdots+16b_2b_4-\frac{g_0}{12}Z^3&=0\,.\end{aligned} \tag{2.9}$$

These equations involve the bare quantities g_0, m_0, Z and $\mathcal{E}_v$ as well as the coefficients b. The second class consists of all the other equations, and these involve only the coefficients b.

$$\begin{aligned}\frac{56b_8}{\pi s}+\frac{2b_{8,2}}{3\pi s^3}+\cdots+24b_2b_6+16b_4^2&=0\,,\\ \frac{12b_{6,2}}{\pi s}+\cdots+16b_2b_{4,2}+48b_{2,2}b_4&=0\,;\end{aligned} \tag{2.10}$$

$$\cdots$$
$$\cdots \tag{2.11}$$

We begin by solving this second class for the b coefficients other than b_2, b_4 and $b_{2,2}$ in terms of g, μ and b_2 using (2.7). Expanding each coefficient as a power series, $b_n=\sum_{r=0..\infty}b_n^r/s^r$, and then equating the coefficients of powers of $1/s$ to zero in each equation gives further equations that can be easily solved in MAPLE to high order. An important feature is that they form a closed system which is easily solved order by order in $1/s$. Consider the equation corresponding to the coefficient of a given local term involving

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F powers of the field and a total of D derivatives. The equation obtained by equating to zero the coefficient of s^{-p} takes the form:

$$\sum_{D'=D}^{D+p-1} A_{F,D,r}^{p,D',r'}\; b_{F+2,D',r'}^{p-1+D-D'} + b_2\, b_{F,D,r}^{p} +$$
$$+ \sum_{D''=0,\,F'=4,\,p'=0\,,r'\,,r''}^{D''=D,\,F'=F-4,\,p'=p} C_{F,D,r}^{p,D'',F',p',r',r''}\; b_{F',D'',r'}^{p'}\; b_{F+2-F',D-D'',r''}^{p-p'} = 0 \tag{2.12}$$

where A and C are known numerical coefficients. This can be solved for $b_{F,D,r}^{p}$ in terms of the coefficients $\{b_{F',D',r'}^{p'}\}$ where $F'+2D'+2p' \le F+2D+2p \equiv Q$ and either $F'=F$, $p'<p$ or $F'<F$, $p'\le p$, $D'\le D$.

$$b_{F,D,r}^{p} = -\frac{1}{b_2}\Bigg(\sum_{D'=D}^{D+p-1} A_{F,D,r}^{p,D',r'}\; b_{F+2,D',r'}^{p-1+D-D'} +$$
$$+ \sum_{D''=0,\,F'=4,\,p'=0\,,r'\,,r''}^{D''=D,\,F'=F-4,\,p'=p} C_{F,D,r}^{p,D'',F',p',r',r''}\; b_{F',D'',r'}^{p'}\; b_{F+2-F',D-D'',r''}^{p-p'}\Bigg). \tag{2.13}$$

To obtain $b_{F,D,r}^{p}$ explicitly we can build up the solutions starting with $p=0$. Given that the renormalisation conditions fix b_4^0 and $b_{2,2}^0$ we begin by calculating b_6^0, $b_{2,4}^0$ and $b_{4,2}^0$ and then use these results to calculate b_8^0, $b_{4,4}^0$, $b_{6,2}^0$ and $b_{2,6}^0$. From these we compute more coefficients until all the $\{b_{F',D'}^0\}$ are known with $F'+2D'\le Q$. Then we move on to $\{b_{F',D'}^1\}$ by first using the results obtained to compute b_6^1, $b_{2,4}^1$ and $b_{4,2}^1$. From these we obtain b_8^1, $b_{4,4}^1$, $b_{6,2}^1$ and $b_{2,6}^1$ and continue the process until all the $\{b_{F',D'}^1\}$ are known with $F'+2D'+2\le Q$, and from these we can compute $\{b_{F',D'}^2\}$ and so on. Although this appears complicated in practice it is straightforward. For example (2.10) gives to lowest order in $1/s$

$$24b_2 b_6^0 + \frac{(g\mu)^2}{4} = 0 \Rightarrow b_6^0 = -\frac{(g\mu)^2}{96b_2}, \qquad 16b_2 b_{4,2}^0 + \frac{3g}{2} = 0 \Rightarrow b_{4,2}^0 = -\frac{3g}{32b_2}. \tag{2.14}$$

Once the $\{b_{F,D}^p\}$ are known substituting the solutions into the first class of equations yields series for the bare quantities. Thus the first of (2.9) gives

$$Z^2 = -\frac{2b_2}{\mu} - \frac{3g}{16b_2\pi s} + \cdots. \tag{2.15}$$

Having obtained the b-coefficients and Z, m_0 and g_0 we solve (2.6) to obtain the c-coefficients and the energy eigenvalue in terms of μ, g and b_2 again as power series in $1/s$. Finally, given that the energy eigenvalue equals μ we can invert the series to obtain b_2 in terms of g and μ, and substitute this into all the other series expansions to express all quantities as power series in $1/s$ depending on g and μ. In the appendix we give the expansions[1] for g_0, and Z up to order $1/s^{17}$ which requires taking terms with $F+2D$ up to 40 in the local expansions (2.2) and (2.5), of which there are about 625 and 540 respectively.

[1]We note in passing that such series could be obtained from standard Feynman diagram expanded in powers of $1/s$, for large s. Given that these expressions contain powers of the coupling to order g^{17} and g^{18} this would require working to 17 loops in conventional perturbation theory.

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These expansions can only be expected to converge for large values of s, if at all, but the continuum limit requires taking s to zero. In the next section we will show how a version of the Borel transform can be used to extract the small s behaviour from these series, and then in section 4 we will apply this to the bare quantities to extract an anomalous scaling dimension which we can test against the exact result for the Ising model.

3. Small s from large s via Borel

The functional integral representation of wave functionals described in section 4 can be expanded in terms of Feynman diagrams. For massive field theories Feynman diagrams are analytic in the cut-off s for large enough s, although for small s they have singularities in the left-half-plane. After renormalisation they have finite limits both as $s \to 0$ and as $s \to \infty$ (where they tend to their tree-level values). Assuming that these properties hold beyond perturbation theory we will exploit this analyticity to extract the small-s behaviour from our series expansions.

As an example, consider a function $f(s)$ that has the assumed analyticity properties of wave-functionals, namely it is analytic for large s, and in the right-half-plane, with finite limits as s approaches the origin and infinity. For large s it will have a Laurent expansion containing only non-positive powers of s, but suppose that we only know the first few terms, $A_N(s) = \sum_{n=0..N} a_n/s^n$, from which we wish to estimate the small s behaviour. We can obtain this from the integral

$$I_\lambda^N = \frac{1}{2\pi i}\int_C ds \frac{e^{\lambda(s-s_0)}}{s-s_0} A_N(s) \tag{3.1}$$

where C is a large circular contour centred on the origin running from just below the negative real axis to just above. This integral is readily computable in terms of the coefficients a_n. Also, since $A_N(s) \approx f(s)$ on C:

$$I_\lambda^N \approx \frac{1}{2\pi i}\int_C ds \frac{e^{\lambda(s-s_0)}}{s-s_0} f(s) \equiv I_\lambda\,. \tag{3.2}$$

If for I_λ the contour C is now collapsed to surround the singularities of the integrand, then the pole at s_0 contributes $f(s_0)$. Poles and cuts lying to the left of s_0 will be suppressed when $\mathrm{Re}(s_0) > 0$, by factors exponential in λ. For example, a pole at $s = s_p$ contributes a piece proportional to $\exp(\lambda(s_p - s_0))$. Similarly the size of the contribution from a contour, C_f, of finite length extending as far to the right as s_c is bounded by

$$\left|\frac{1}{2\pi i}\int_{C_f} ds \frac{e^{\lambda(s-s_0)}}{s-s_0} f(s)\right| < e^{\mathrm{Re}(\lambda(s_c-s_0))}\frac{1}{2\pi}\int_{C_f}\frac{|ds|\,|f(s)|}{|s-s_0|}\,. \tag{3.3}$$

This exponential damping means that for large λ

$$I_\lambda^N \approx f(s_0)\,. \tag{3.4}$$

We cannot take λ arbitrarily large, but only as large as allows I_λ^N to be a good approximation to I_λ, which is the case when I_λ^{N-1} differs by no more than say a few per cent from I_λ^N. If this condition fixes λ to be $\lambda *$ then we arrive at the approximation $f(s_0) \approx I_{\lambda *}^N$

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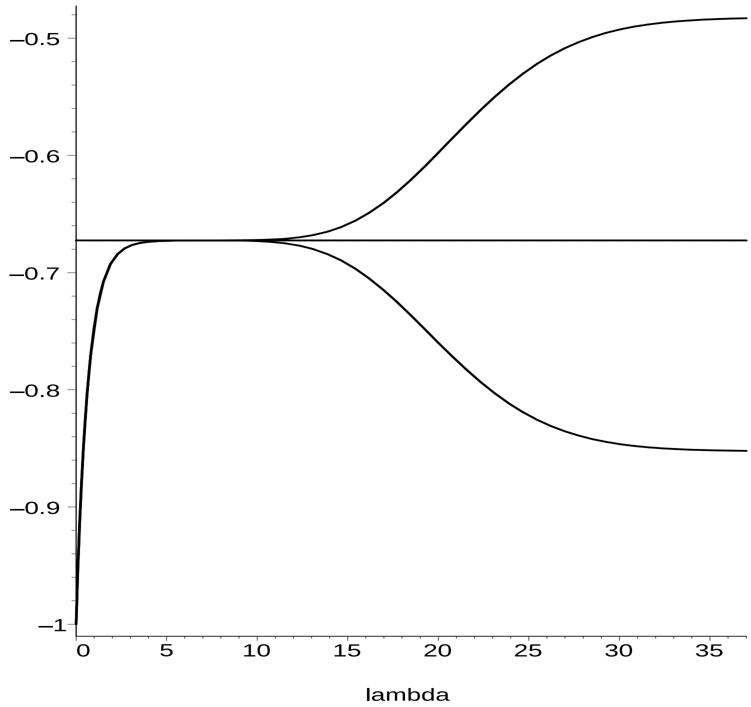

Figure 3: I_λ^{18} (lower curve) and I_λ^{19}.

We will illustrate this with the truncated series $A_N = \sum_{n=0}^{N}(-1/s)^n/(n+1)$. As $N \to \infty$ this series converges for $|s| > 1$ to $f = s\log(s/(s+1))$, which is analytic except for a cut from $s = -1$ to the origin, so

$$I_\lambda = s_0 \log\left(\frac{s_0}{s_0+1}\right) + \int_0^1 dy \frac{y}{y+s_0} e^{-\lambda(y+s_0)} \tag{3.5}$$

from which it is clear that $\lim_{\lambda\to\infty} I_\lambda = f(s_0)$. If we use the series truncated at say nineteen terms then $A_{19}(s)$ itself approximates $f(s)$ to better than one percent for $s > 1.07$. At $s = 1$ the error grows to three and a half per cent and for $|s| < 1$ the error increases rapidly as s gets smaller. Figure 3 plots I_λ^{18} and I_λ^{19} for $s_0 = 0.9$. The two curves agree up to about $\lambda = 9$, and for this value provide a very good approximation to $f(s_0) = -0.6725$, ($I_\lambda^{19} = -0.6724$ and $I_\lambda^{18} = -0.6727$). For larger values the two curves diverge, and approach $A_{19}(0.9) = -0.4823$ and $A_{18}(0.9) = -0.8524$ which are poor approximations to $f(0.9)$. For much smaller values of s_0 we still obtain reasonable results, for example in the case of $s_0 = 0.05$ the two curves diverge at $\lambda \approx 7$ yielding an approximation of $I_\lambda^{19} = -0.2123$ to $f(0.05) = -0.1522$, which is fifty per cent out, but it is amusing to compare this to the value of $A_{19}(0.05) = 0.25 \times 10^{24}$.

We will be principally interested in the value of f at the origin. We can take $s_0 \to 0$ in the general case, but the new feature that arises is that any cut in the left half-plane that ends at the origin will no longer be exponentially suppressed. However, our assumption that $f(s)$ has a finite limit as $s \to 0$ implies that if $f \sim s^\kappa$ then $\kappa > 0$ and this behaviour is power law suppressed. Also the expression for I_λ^N simplifies to $\sum_{n=0..N} a_n \lambda^n/n!$. In our example

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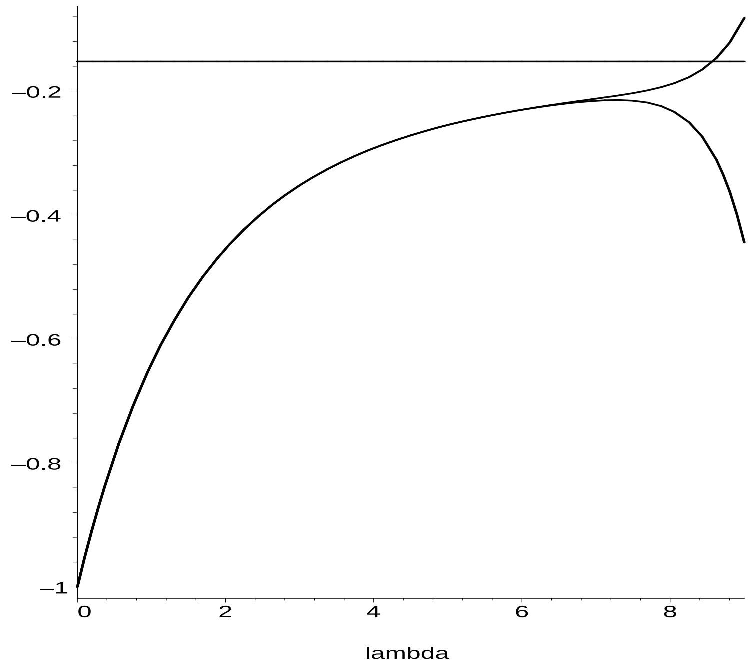

Figure 4: I_λ^{19} (upper curve) and I_λ^{18} for $s \to 0$.

$I_\lambda = e^{-\lambda}/\lambda - 1/\lambda$ and the second term demonstrates this power law suppression. Again I_λ^{19} and I_λ^{18} agree for λ up to about 7 yielding an approximation of -0.1416 to $f(0) = 0$.

This approximation can be greatly improved by changing the complex variable s, for example by considering $f(s^a)$ with $a > 1$ instead of $f(s)$, so that now

$$I_\lambda^N = \sum_{n=0,\ldots,N} \frac{a_n \lambda^{na}}{\Gamma(na+1)} \,. \tag{3.6}$$

Increasing a reduces the size of the last term in I_λ^N in comparison to the penultimate term, because of the Γ function in (3.6), enabling us to take a larger value of λ, however increasing a enlarges the region of non-analyticity. Initially this will result in singularities occurring at values of s with large negative parts, causing oscillations in the plot of I_λ^N against λ. This can be seen in our example, in figure 5.

Maxima appear for $a = 1.4$, 1.5 and 1.6 due to these oscillations. Increasing a still further can cause the singularities to migrate to the right half-plane, spoiling our construction altogether. The position of these dominant singularities can be estimated by studying the shape of the curve, thus the $\exp(-\lambda)$ decay in our example for $a = 1$ can be seen by fitting numerically the curve of I_λ^N to an exponential. To improve our approximation we need to increase a as much as we can without encountering these pathologies. We can do this by taking a as large as possible whilst maintaining I_λ^N as a monotonic function of λ in the region where it approximates $f(0)$, $\lambda < \lambda*$, (we may have to exclude a region close to the origin if I_λ has a turning point not due to a singularity occurring at complex s, this complication can be resolved by studying the N-dependence). So in our example we would

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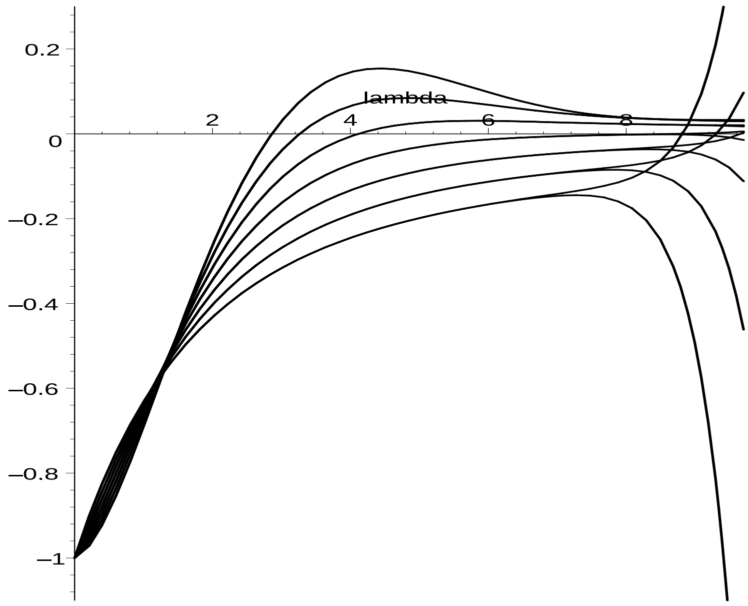

Figure 5: I_λ^{19} and I_λ^{18} for $a = 1$ (lowest curve), $1.1, \ldots, 1.6$.

take $a = 1.348004$ and obtain the approximation 0.013 when we require that I_λ^{18} and I_λ^{19} differ by 0.01 at $\lambda* = 9.77$.

The precision with which we require I_λ^{19} and I_λ^{18} to agree introduces arbitrariness into $\lambda*$, however because we tune a so that I_λ^{19} is quite flat for $\lambda \approx \lambda*$ the estimate we obtain is fairly insensitive to this choice of precision, so for example if we require that I_λ^{19} and I_λ^{18} differ by only 0.001 (instead of 0.01) at $\lambda*$ the estimate changes to 0.0115 even though $\lambda*$ has changed to 8.93. Note that I_λ^{18} is considerably less flat than I_λ^{19} in the vicinity of $\lambda*$. It has a single maximum in this region, whereas the flat portion of the latter turns into two turning points that are close together if a is slightly increased. To minimise arbitrariness we should in general base our estimates on the flatter curve, which means using I_λ^N with N odd in this example.

The integral I_λ is the Borel transform of the function f centred at s_0. Our approach based on studying the large-λ behaviour to reconstruct the original function has the advantage over the usual method of avoiding the need to analytically continue the transform, (for example using Padé approximants), prior to inversion using the Laplace transform.

In our application to the Schrödinger equation s is a sharp momentum cut-off. It has been observed that in the different, but related context, of Polchinski's flow equation that a smooth cut-off increases the rate of convergence of a series solution. The reverse occurs here. Suppose that the series A_N resulted from an integral with a sharp cut-off as might be the case for example for a one-loop Feynman diagram. Thus

$$A_N = \sum_{n=0}^{N} \frac{a_n}{s^n} = \int_0^{1/s} dk \sum_{n=0}^{N} a_n \, n k^{n-1} \, . \tag{3.7}$$

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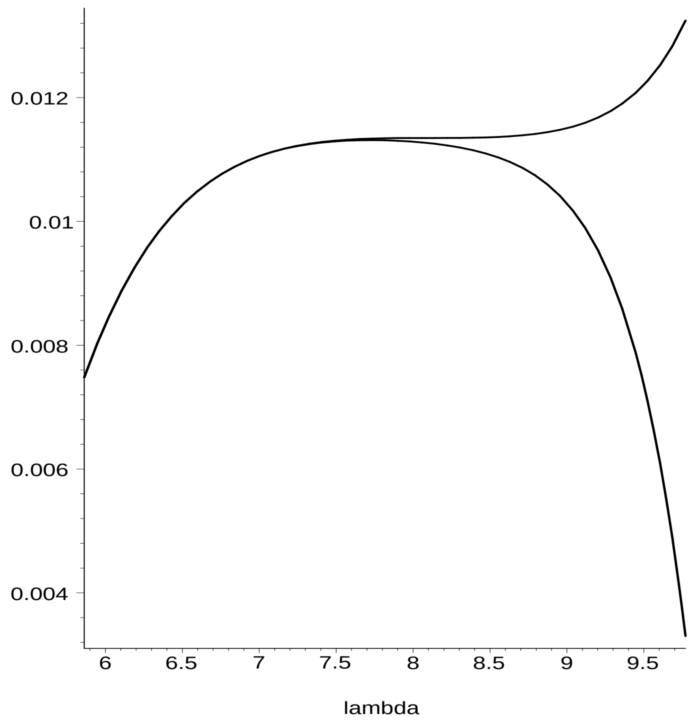

Figure 6: I_λ^{19} (upper curve) and I_λ^{18} for $a = 1.348004$ in the vicinity of $\lambda*$.

Our transform I_λ^N is $\sum_{n=0..N} a_n \lambda^n/n!$ as before, but if we replace the sharp cut-off by a gaussian one we obtain an alternative to A_N

$$\int_0^\infty dk \sum_{n=0}^{N} a_n \, n k^{n-1} \, e^{-s^2k^2} = \frac{1}{2}\sum_{n=0}^{N} \Gamma\left(\frac{n}{2}\right)\frac{a_n}{s^n} \tag{3.8}$$

with a transform $\sum_{n=0..N} \Gamma(n/2)\, a_n \lambda^n/(2n!)$ which clearly converges much more slowly (if at all) due to the $\Gamma(n/2)$ in the numerator. That this is the reverse to what occurs in the Polchinski flow equation in [33] is because in that application the corresponding expansions involve positive powers of s rather than negative ones so that $\Gamma(n/2)$ appears instead in the denominator and enhances convergence.

4. Surface critical scaling in the Ising model

φ^4 theory is the Landau-Ginzburg theory that describes the critical behaviour of the Ising model [34, 35]. The vacuum wave-functional has a functional integral representation as a euclidean field theory with a boundary corresponding to the quantisation surface

$$\Psi[\varphi] = \int \mathcal{D}\phi \exp\left(-\int_{-\infty}^{0} dt \int_{-\infty}^{\infty} dx \left(\frac{1}{2}\left(\dot\phi^2 + \phi'^2 + m_0^2\,\phi^2\right) + \frac{g_0}{4!}\,\phi^4\right) + \int_{-\infty}^{\infty} dx \varphi_0\, \dot\phi(x,0)\right) \tag{4.1}$$

where the ϕ satisfies Dirichlet boundary conditions $\phi(x,0) = 0$, and a regulator is imposed (which we take as the same momentum cut-off as before). φ_0 appears as a source term

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coupled to $\dot{\phi}$ so that functional differentiation results in an insertion of $\dot{\phi}$. For a particular value of g the $\mu \to 0$ limit of Ψ models the critical behaviour of the two-dimensional Ising model with a boundary on which the spins $\sigma \sim \phi$ obey Dirichlet boundary conditions, this is the Ordinary Transition. The correlator $\langle \dot{\sigma}(0,0)\, \dot{\sigma}(x,0) \rangle$ can be computed from the exact Onsager solution or from the conformal field theory four-point function using the method of images and shown to depend on $|x|$ as $1/|x|$, [31]. So in the Conformal Field Theory limit of φ^4 that part of W quadratic in the field,$\int dx\, dy\, \varphi_0(x)\, \varphi_0(y)\, \Gamma_0(x-y) = \int dx\, dy\, \varphi(x)\, \varphi(y)\, \Gamma(x-y)$, should have $\Gamma \propto 1/|x-y|$.

We have studied massive φ^4 theory, but the Renormalisation Group provides a means of extracting the massless behaviour [34, 35]. It starts from the observation that at short distances the functional integral (4.1) expressed as it is in terms of bare quantities is approximately independent of m_0. This is based on a perturbative analysis, because in terms of Feynman diagrams differentiating Ψ with respect to m_0 introduces a ϕ^2 insertion in the bulk involving two propagators that are suppressed at large momentum. Now, from the functional integral representation Γ_0 is a function of the bare variables g_0, m_0 and the cut-off and is related by wave-function renormalisation to Γ which is given by our solution in terms of μ and g. By dimensional analysis Z depends on μ and s only in the combination μs whilst Γ depends on $x-y$ as $F(\mu(x-y),g)/|x-y|^2$ with F a dimensionless function, so

$$\Gamma_0(x-y, g_0, m_0, s) = Z^{-1}(\mu s, g) \frac{F(\mu(x-y), g)}{|x-y|^2} \,. \tag{4.2}$$

Since Γ_0 can be taken to be independent of m_0 for small $|x-y|$ the left-hand-side does not change if m_0 is varied whilst g_0 and s are held fixed. As m_0 varies μ and g will vary together, so that g can be thought of as a function of μ with a β-function, $\beta = dg/d\log\mu = \beta(g)$. Since Γ_0 does not change if we vary μ and g from μ_1 and g_1 to μ_2 and $g_2 = g(\mu_1, \mu_2, g_1)$we obtain

$$F(\mu_2(x-y), g_2) = \frac{Z(\mu_2 s, g(\mu_1, \mu_2, g_1))}{Z(\mu_1 s, g_1)} F(\mu_1(x-y), g_1) \,. \tag{4.3}$$

Now $\gamma \equiv d\log Z/d\log\mu$ is a function of g, so

$$\frac{Z(\mu_2 s, g(\mu_1, \mu_2, g_1))}{Z(\mu_1 s, g_1)} = \exp \int_{\mu_1}^{\mu_2} \frac{d\mu}{\mu}\gamma = \exp \int_{g_2}^{g_1} \frac{dg}{\beta}\gamma \,. \tag{4.4}$$

If g_* is a zero of the β-function and if we assume that γ is continuous then for g_1, g_2 close to g_* (4.4) behaves like

$$\exp \int_{g_2}^{g_1} \frac{dg}{(g-g_*)\, \beta'(g_*)} \gamma(g_*) \approx \left(\frac{\mu_2}{\mu_1}\right)^{\gamma(g_*)} \,. \tag{4.5}$$

If in addition we assume that $F(x,g)$ is continuous near g_* then the limit as $g_1 \to g_*$ of (4.3) is

$$F(\mu_2(x-y), g_*) = \left(\frac{\mu_2}{\mu_1}\right)^{\gamma(g_*)} F(\mu_1(x-y), g_*) \tag{4.6}$$

which implies the homogeneous dependence on $x-y$: $F \sim |x-y|^{\gamma(g_*)}$ that is characteristic of a Conformal Field Theory.

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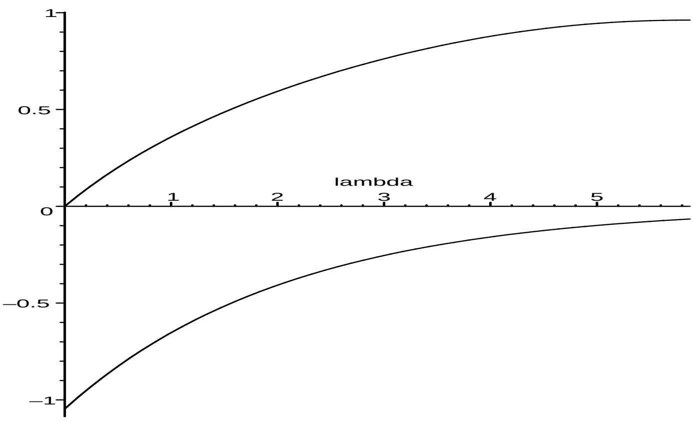

Figure 7: I_λ^{17} (upper curve) and $\exp(m\lambda + c)/m$ approximating the error.

The correspondence betwen the Ising model and φ^4 theory requires that we should be able to obtain $\gamma(g_*) = 1$ from our solution. The series given in the appendix express g_0 and Z as functions of μ, s and g. Differentiating these with respect to $\log \mu$ whilst keeping g_0 and s fixed gives

$$0 = \frac{\partial g_0}{\partial \log \mu} + \beta \frac{\partial g_0}{\partial g}, \qquad \gamma Z = \frac{\partial Z}{\partial \log \mu} + \beta \frac{\partial Z}{\partial g}. \tag{4.7}$$

From which β and γ are obtained as power series in $1/s$ with the results also given in the appendix up to order $1/s^{17}$. We have calculated these functions for a variety of values of g using the transformation of section 3 to extract the $s \to 0$ limit. The results are shown in figure 2 and demonstrate that the β-function has a zero at g roughly equal to 2 where γ is approximately unity in agreement with the Ising model result. Studying the region in the vicinity of $g = 2$ leads to values of 1.955 for the zero and 0.961 for the corresponding value of γ.

Comparison with the exact result shows a departure from it by four per cent, but we can estimate the error *a priori* by examining the shape of I_λ^N. For small λ the error is mainly composed of exponential terms that are suppressed as λ increases, we can estimate the dominant term by fitting $dI_\lambda^N/d\lambda$ to an expression of the form $\exp(m\lambda+c)$ for $0 < \lambda < 0.9\lambda*$ (the region is only taken up to $0.9\lambda*$ rather than λ because the $dI_\lambda^{17}/d\lambda$ and $dI_\lambda^{16}/d\lambda$ diverge for smaller λ). In figure 7 we plot I_λ^{17} and $\exp(m\lambda + c)/m$ which shows that the exponential terms account for about six per cent of the value of I_λ^N, this is of course only a rough estimate.

This approximation of the anomalous dimension depends on the number of terms we keep in the series expansions, which in turn determines the type of terms we need to retain in our local expansions of W and U. So far we have worked to order $1/s^n$ with $n = 17$ which has required including terms with F fields and D derivatives such that $F + 2D \leq 40$. Since the true continuum limit should be obtained as $n \to \infty$ it is worthwhile to see how the results vary with n. In figure 8 we plot the approximations obtained by keeping n terms against $1/n$ for $n = 17, 15, 13, 11, 9$. We also show the exact result as though it would occur at $n = \infty$, to guide the eye. The plot shows roughly linear behaviour, and when

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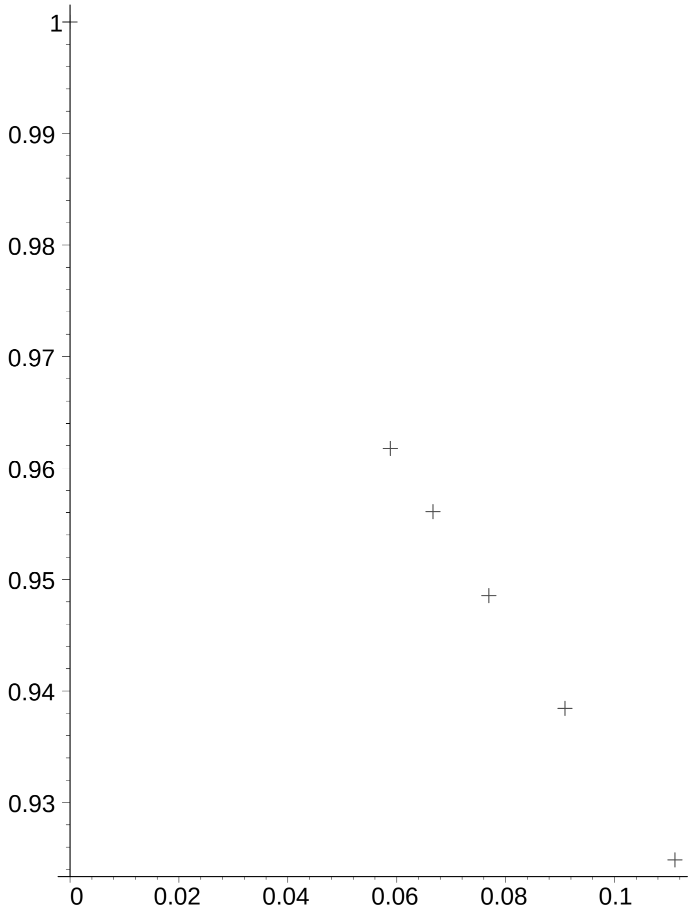

Figure 8: The approximation to η for $n = 17, 15, 13, 11, 9$.

a straight-line fit through these results is extrapolated we obtain a value of 1.003 for the anomalous dimension, which is in very good agreement with the exact Ising Model result.

We can compare this result with the predictions of one-loop perturbation theory. The vacuum functional was computed in the semi-classical approximation in [36] (the perturbative treatment of the three-dimensional case is given in [37]) where the coefficients of φ_0^4 and $\varphi_0'^2$ in W were found to be $-g_0/(96M) + 3.973.10^{-4} g_0^3/M^3$ and $-1/(4M) + g_0/(48\pi M)$, respectively, where M is related to the particle mass by $M = \mu - g_0/(8\pi\mu)$. These imply that at one-loop order $\beta = -2g + 0.617g^2$ and $\gamma = g/\pi$, so that $g_* \approx 3.242$ and $\gamma(g_*) \approx 1.032$, which is slightly more accurate than our estimate based on series with 17 terms, but considerably less accurate than the extrapolated value of 1.003.

The expansions in the appendix can be re-ordered to yield estimates for the coefficients of the loop expansion. For example, if we retain only those terms up to and including second order in g in the beta-function we obtain

$$\beta \approx -2g + g^2 \left(\frac{15}{8} \frac{1}{\pi \mu s} - \frac{65}{72} \frac{1}{\mu^3 s^3 \pi} + \frac{3773}{5760} \frac{1}{\mu^5 s^5 \pi} - \frac{179}{336} \frac{1}{\mu^7 s^7 \pi} + \right.$$
$$\left. + \frac{341759}{746496} \frac{1}{\mu^9 s^9 \pi} - \frac{1112423}{2737152} \frac{1}{\mu^{11} s^{11} \pi} + \frac{38151475}{103514112} \frac{1}{\mu^{13} s^{13} \pi} - \right.$$

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$$- \frac{182364253}{537477120} \frac{1}{\mu^{15} s^{15} \pi} + \frac{9235090201}{29238755328} \frac{1}{\mu^{17} s^{17} \pi} \Bigg) . \tag{4.8}$$

Using the Borel resummation technique to find the $s \to 0$ limit of the coefficient of g^2 gives 0.633 which is in reasonable agreement with the one-loop value 0.617 given that this series has so few terms in comparison with the previous series we have considered. Similarly the one-loop contribution to γ is approximated by

$$\begin{aligned} \gamma \approx g \Bigg(& \frac{9}{8} \frac{1}{\pi \mu s} - \frac{5}{8} \frac{1}{\mu^3 s^3 \pi} + \frac{63}{128} \frac{1}{\mu^5 s^5 \pi} - \frac{27}{64} \frac{1}{\mu^7 s^7 \pi} + \frac{385}{1024} \frac{1}{\mu^9 s^9 \pi} - \\ & - \frac{351}{1024} \frac{1}{\mu^{11} s^{11} \pi} + \frac{10395}{32768} \frac{1}{\mu^{13} s^{13} \pi} - \frac{2431}{8192} \frac{1}{\mu^{15} s^{15} \pi} + \frac{73359}{262144} \frac{1}{\mu^{17} s^{17} \pi} \Bigg) . \end{aligned} \tag{4.9}$$

Resummation yields the value $0.336g$ which should be compared with $g/\pi \approx 0.318\, g$.

We can go further and estimate the two-loop contribution to the beta-function from the series

$$\begin{aligned} g^3 \Bigg(& - \frac{9}{2} \frac{1}{\pi \mu^2 s^2} + \frac{65}{18} \frac{1}{\pi \mu^4 s^4} + \frac{5975}{10368} \frac{1}{\pi \mu^6 s^6} - \frac{1082059}{60480} \frac{1}{\pi \mu^8 s^8} + \\ & + \frac{637469833}{7464960} \frac{1}{\pi \mu^{10} s^{10}} - \frac{73790276321}{215550720} \frac{1}{\pi \mu^{12} s^{12}} + \frac{52328568942469}{39852933120} \frac{1}{\pi \mu^{14} s^{14}} - \\ & - \frac{1607134369983293}{322808758272} \frac{1}{\pi \mu^{16} s^{16}} \Bigg) \end{aligned} \tag{4.10}$$

which gives $-0.813g^2$ on resummation, so to two-loop order the β-function ceases to have a zero. The disappearance (and reappearance) of the zero in successive orders of perturbation theory is encountered elsewhere and requires a skilful handling of the perturbation expansion, for example with the use of Padé-Borel resummation in g, [37]. This is to be contrasted with the results from the Schrödinger equation for which there is always a zero in the β-function although its position is corrected at each increasing order of the approximation.

5. Conclusions

We have solved the functional Schrödinger equation for the vacuum of φ^4 theory and for a state describing a stationary particle by expanding these states in terms of local functionals. The solutions were obtained as power series in $1/s$, where s is a short-distance cut-off. The $s \to 0$ limits of these series were found using a variant of Borel re-summation that avoids the use of analytic continuation. A standard Renormalisation Group argument was used to compute an anomalous scaling dimension in the massless theory. Extrapolating in the number of terms of the series gave the result 1.003 for this scaling dimension which agrees well with the exact result of unity obtained from the conformal field theory treatment of the surface critical scaling of the Ising model at the Ordinary Transition.

We have only developed the solution to the point where we could make the comparison with the Ising model. Further work remains to be done to describe the particle dynamics in this approach, for example by checking Lorentz invariance and constructing an S-matrix.

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Neither have we computed inner products of states, although their form is determined by the Hermiticity of the hamiltonian and they should also be calculable as re-summed expansions in $1/s$.

We concentrated on two dimensional scalar theory as this allowed the series expansions to be obtained using only modest computing resources. Similar resources should be sufficient to study other models such as Landau-Ginzburg theories with higher order polynomial interactions and integrable models with exponential interactions where the wealth of exact results would provide useful tests of the method. There is nothing intrinsic to the approach that limits it to two dimensions, so it should be possible to apply it to critical phenomena in 3 dimensions and gauge theories in three and four. Indeed the method was motivated by the desire to systematise a heuristic model of the Yang-Mills vacuum that gave an accurate formula for the string tension in three dimensions. Applying it to that case to compute series with large numbers of terms would require some effort, so we cannot at the moment explain why the formula for the string-tension agrees accurately with Monte-Carlo results, despite being based on just the leading term of the local expansion. However we can understand why the corresponding calculation in four-dimensions appears to give a string-tension that diverges with the cut-off: this is just the first term in a power series that should re-sum to a finite result. As in the case of scalar field theory standard perturbation theory is obtained by organising the local expansion in powers of the coupling. In this context it has already been shown how the usual one-loop beta-function for Yang-Mills theory is obtained in this approach [38].

A. Solutions

The solutions listed below for the series expansions of Z, g_0, β and γ were computed up to order $1/s^{17}$ using MAPLE, working with rational numbers. The exact values for the cofficients have been evaluated to ten significant figures to save space.

Solving the Schrödinger equation yields the series expansion of the wave-function renormalisation, $Z = \sum_n z_n/(\mu s)^n$, with the first 18 coefficients given by

$$\begin{aligned}
z_0 &= 1.0\\
z_1 &= -0.1193662073\, g\\
z_2 &= 0.05066059180\, g^2\\
z_3 &= 0.03978873576\, g - 0.07596748147\, g^3\\
z_4 &= 0.1658372186\, g^4 - 0.04239306467\, g^2\\
z_5 &= 0.1322116938\, g^3 - 0.4702098769\, g^5 - 0.02238116387\, g\\
z_6 &= 1.600512277\, g^6 - 0.4288302602\, g^4 + 0.02320184743\, g^2\\
z_7 &= 0.01492077591\, g + 1.643970766\, g^5 - 6.280656209\, g^7 - 0.2150394698\, g^3\\
z_8 &= -7.118564002\, g^6 + 27.72021990\, g^8 + 0.03759400389\, g^2 + 0.9449811249\, g^4\\
z_9 &= -0.01087973244\, g - 4.714821300\, g^5 + 34.16960776\, g^7 -\\
&\quad -135.3903272\, g^9 + 0.3954218093\, g^3
\end{aligned}$$

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$$
\begin{aligned}
z_{10} &= 723.5185086\, g^{10} + 25.14159983\, g^{6} - 179.4495800\, g^{8} - \\
&\quad -0.2452396545\, g^{2} - 2.090535257\, g^{4} \\
z_{11} &= 0.008392936450\, g - 4195.152125\, g^{11} + 14.06398176\, g^{5} - \\
&\quad -143.8822880\, g^{7} + 1021.740867\, g^{9} - 0.9435065158\, g^{3} \\
z_{12} &= 26222.36179\, g^{12} - 6265.013026\, g^{10} - 92.19320030\, g^{6} + \\
&\quad +880.6482723\, g^{8} + 0.9593860756\, g^{2} + 3.790793362\, g^{4} \\
z_{13} &= -0.006731834444\, g - 175775.0461\, g^{13} + 41156.19353\, g^{11} - 45.34168199\, g^{5} + \\
&\quad +626.1457656\, g^{7} - 5746.450353\, g^{9} + 3.174250642\, g^{3} \\
z_{14} &= -288442.7166\, g^{12} + 1258090.875\, g^{14} + 39859.60610\, g^{10} + 375.4551915\, g^{6} - \\
&\quad -4438.133512\, g^{8} - 3.420232517\, g^{2} + 9.605144838\, g^{4} \\
z_{15} &= 0.005556434780\, g + 2149122.481\, g^{13} - 293133.1749\, g^{11} + 138.6466218\, g^{5} - \\
&\quad -3087.933824\, g^{7} + 32967.21250\, g^{9} - 14.23407683\, g^{3} - 9578735.713\, g^{15} \\
z_{16} &= 2280136.160\, g^{12} - 16970850.80\, g^{14} - 256999.1253\, g^{10} - \\
&\quad -1543.206840\, g^{6} + 25586.11980\, g^{8} + 11.94349655\, g^{2} - \\
&\quad -272.9892577\, g^{4} + 77323626.71\, g^{16} \\
z_{17} &= -0.004688241844\, g + 75.31011486\, g^{3} + 16703.89651\, g^{7} - \\
&\quad -141.2669974\, g^{5} + 141642086.6\, g^{15} - 18718142.21\, g^{13} + \\
&\quad +2102889.466\, g^{11} - 217344.7165\, g^{9} - 659842155.7\, g^{17}\,.
\end{aligned}
$$

Solving the Schrödinger equation yields the series expansion of the bare-coupling, $g_0 = \mu^2 \sum_n \kappa_n/(\mu s)^n$, with the first 18 coefficients given by

$$
\begin{aligned}
\kappa_0 &= 1.0\, g \\
\kappa_1 &= 0.1989436788\, g^{2} \\
\kappa_2 &= -0.03641230036\, g^{3} \\
\kappa_3 &= 0.08207763547\, g^{4} - 0.05747261834\, g^{2} \\
\kappa_4 &= -0.1813715346\, g^{5} + 0.02286762825\, g^{3} \\
\kappa_5 &= 0.5276961593\, g^{6} - 0.1344338864\, g^{4} + 0.02978628969\, g^{2} \\
\kappa_6 &= -1.827461455\, g^{7} + 0.4342310123\, g^{5} + 0.02099375315\, g^{3} \\
\kappa_7 &= 0.2040031164\, g^{4} - 0.01884175582\, g^{2} - 1.725204731\, g^{6} + 7.268618386\, g^{8} \\
\kappa_8 &= -0.8376851525\, g^{5} - 0.1619837103\, g^{3} + 7.650541972\, g^{7} - 32.43432692\, g^{9} \\
\kappa_9 &= -0.3373967928\, g^{4} + 0.01324798773\, g^{2} + 4.478764811\, g^{6} - \\
&\quad -37.43268679\, g^{8} + 159.8791617\, g^{10} \\
\kappa_{10} &= 1.190013912\, g^{5} + 0.6263468208\, g^{3} - 24.78826072\, g^{7} + \\
&\quad +199.6885243\, g^{9} - 861.1578312\, g^{11} \\
\kappa_{11} &= 0.7212680080\, g^{4} - 0.009951254171\, g^{2} - 10.64530019\, g^{6} + 146.1049618\, g^{8} - \\
&\quad -1152.010150\, g^{10} + 5027.712587\, g^{12}
\end{aligned}
$$

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$$
\begin{aligned}
\kappa_{12} &= 3.152841368\, g^5 - 2.174458226\, g^3 + 76.02682717\, g^7 - 915.5371834\, g^9 + \\
&\quad +7143.555824\, g^{11} - 31617.81483\, g^{13} \\
\kappa_{13} &= -2.418827270\, g^4 + 0.007821150456\, g^2 + 13.86178637\, g^6 - 549.6708328\, g^8 + \\
&\quad +6091.031330\, g^{10} - 47386.89418\, g^{12} + 213090.6078\, g^{14} \\
\kappa_{14} &= -68.20413916\, g^5 + 7.396453034\, g^3 - 178.3373919\, g^7 + 4076.316308\, g^9 - \\
&\quad -42945.16713\, g^{11} + 334961.5945\, g^{13} - 1532576.001\, g^{15} \\
\kappa_{15} &= 11.95109263\, g^4 - 0.006353030446\, g^2 + 151.2121465\, g^6 + 1867.345135\, g^8 - \\
&\quad -31358.18697\, g^{10} + 320273.7334\, g^{12} - 2514689.736\, g^{14} + 11719597.0\, g^{16} \\
\kappa_{16} &= 819.5644108\, g^5 - 25.22638048\, g^3 - 790.3286415\, g^7 - 17670.92565\, g^9 + \\
&\quad +251437.2784\, g^{11} - 2521699.943\, g^{13} + 19992199.42\, g^{15} - 94979792.35\, g^{17} \\
\kappa_{17} &= -167874034.0\, g^{16} - 2682.130056\, g^6 + 0.005291499834\, g^2 - \\
&\quad -69.91144120\, g^4 + 436.3776317\, g^8 + 163982.6175\, g^{10} - \\
&\quad -2105884.503\, g^{12} + 20923210.23\, g^{14} + 813423914.7\, g^{18}\,.
\end{aligned}
$$

The solutions for Z and g_0 lead to the series expansion of the β-function, $\beta = \sum_n \beta_n/(\mu s)^n$, with the first 18 coefficients given by

$$
\begin{aligned}
\beta_0 &= -2.0\, g \\
\beta_1 &= 0.5968310364\, g^2 \\
\beta_2 &= -0.4559453262\, g^3 \\
\beta_3 &= 0.9853095730\, g^4 - 0.2873630916\, g^2 \\
\beta_4 &= 0.3658820519\, g^3 - 2.814252613\, g^5 \\
\beta_5 &= 0.2085040278\, g^2 + 9.833762561\, g^6 - 1.749096318\, g^4 \\
\beta_6 &= 0.05839063195\, g^3 - 39.74109388\, g^7 + 7.374962709\, g^5 \\
\beta_7 &= -0.1695758024\, g^2 + 180.4235628\, g^8 - 34.76674176\, g^6 + 2.702926953\, g^4 \\
\beta_8 &= -903.9912677\, g^9 - 1.812756260\, g^3 + 178.4867048\, g^7 - 15.43817775\, g^5 \\
\beta_9 &= 0.1457278651\, g^2 - 990.1865501\, g^8 + 96.47016296\, g^6 - \\
&\quad -4.096644239\, g^4 + 4940.346009\, g^{10} \\
\beta_{10} &= 5898.620256\, g^9 - 29206.19354\, g^{11} + 8.652316689\, g^3 - \\
&\quad -615.2794259\, g^7 + 24.26339449\, g^5 \\
\beta_{11} &= -0.1293663043\, g^2 + 4091.802192\, g^8 - 241.2179221\, g^6 + \\
&\quad +7.386237270\, g^4 - 37554.52475\, g^{10} + 185620.8411\, g^{12} \\
\beta_{12} &= -28511.70908\, g^9 + 254595.6560\, g^{11} - 34.68565605\, g^3 + \\
&\quad +1992.597606\, g^7 + 59.07548891\, g^5 - 1262105.326\, g^{13} \\
\beta_{13} &= 9144064.399\, g^{14} + 0.1173172568\, g^2 - 16205.08595\, g^8 + 289.2141186\, g^6 - \\
&\quad -26.54957599\, g^4 + 208570.6632\, g^{10} - 1832311.357\, g^{12} \\
\beta_{14} &= 133241.9688\, g^9 - 1602568.994\, g^{11} + 133.0389542\, g^3 - 4843.070749\, g^7 -
\end{aligned}
$$

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$$
\begin{aligned}
& -1489.424050\, g^5 + 13962796.56\, g^{13} - 70350835.71\, g^{15} \\
\beta_{15} &= -112396967.2\, g^{14} - 0.1080015175\, g^2 + 57364.96045\, g^8 + 4603.125796\, g^6 + \\
& +573028054.8\, g^{16} + 177.9010353\, g^4 - 1123852.943\, g^{10} + 12930212.94\, g^{12} \\
\beta_{16} &= -602711.1263\, g^9 + 9794809.468\, g^{11} - 504.4372323\, g^3 - 24799.33347\, g^7 + \\
& +19590.50509\, g^5 - 109478969.7\, g^{13} + +953696332.6\, g^{15} - 4928124568.0\, g^{17} \\
\beta_{17} &= 44638767880.0\, g^{18} + 6126484.834\, g^{10} - 82996.74004\, g^6 - \\
& -1336.249746\, g^4 + 0.1005384968\, g^2 + 971831675.9\, g^{14} + \\
& +27515.67004\, g^8 - 88542747.03\, g^{12} - 8512571797.0\, g^{16} \,.
\end{aligned}
$$

The solutions for Z and g_0 lead to the series expansion of the the logarithmic derivative of Z, $\gamma = \sum_n \gamma_n/(\mu s)^n$, with the first 17 coefficients given by

$$
\begin{aligned}
\gamma_1 &= 0.3580986219\, g \\
\gamma_2 &= -0.3324601337\, g^2 \\
\gamma_3 &= -0.1989436788\, g + 0.7407774312\, g^3 \\
\gamma_4 &= -2.157405193\, g^4 + 0.3591976683\, g^2 \\
\gamma_5 &= -1.514499301\, g^3 + 7.609034060\, g^5 + 0.1566681471\, g \\
\gamma_6 &= -30.90102012\, g^6 + 6.324551298\, g^4 - 0.2470671327\, g^2 \\
\gamma_7 &= 2.810587640\, g^3 + 140.6420098\, g^7 - 29.47246351\, g^5 - 0.1342869832\, g \\
\gamma_8 &= -15.56377334\, g^4 - 0.4393122678\, g^2 - 705.4720543\, g^8 + 149.9640642\, g^6 \\
\gamma_9 &= 3856.625518\, g^9 + 0.1196770568\, g - 5.755944500\, g^3 \\
& -825.7869756\, g^7 + 92.59293353\, g^5 \\
\gamma_{10} &= 3.423764699\, g^2 + 4887.996878\, g^8 - 573.0603460\, g^6 + \\
& +38.04564185\, g^4 - 22794.88025\, g^{10} \\
\gamma_{11} &= -0.1091081738\, g - 30947.00771\, g^9 + 144798.2478\, g^{11} + \\
& +15.15984462\, g^3 + 3727.952024\, g^7 - 299.6137729\, g^5 \\
\gamma_{12} &= 208764.5292\, g^{10} - 983835.0583\, g^{12} - 15.34219367\, g^2 - \\
& -25536.87550\, g^8 + 2257.479359\, g^6 - 75.61099893\, g^4 \\
\gamma_{13} &= 0.1009775167\, g + 184253.1860\, g^9 - 1495814.945\, g^{11} - 56.87562475\, g^3 - \\
& -17294.46868\, g^7 + 1035.456051\, g^5 + 7122090.122\, g^{13} \\
\gamma_{14} &= 61.55740610\, g^2 + 136372.4752\, g^8 - 9808.072105\, g^6 - 54746191.69\, g^{14} - \\
& -213.8081010\, g^4 - 1399600.487\, g^{10} + 11353026.88\, g^{12} \\
\gamma_{15} &= -0.09445939125\, g - 3285.663980\, g^5 - 91057448.51\, g^{13} + 445520492.1\, g^{15} - \\
& -1114679.686\, g^9 + 11183177.87\, g^{11} + 286.2007410\, g^3 + 90432.28371\, g^7 \\
\gamma_{16} &= 42485.83958\, g^6 + 6557.967853\, g^4 + 9477782.019\, g^{10} - 93893690.17\, g^{12} - \\
& -3828085628.0\, g^{16} + 770074284.4\, g^{14} - 238.8640816\, g^2 - 829515.1962\, g^8 \\
\gamma_{17} &= -514466.9761\, g^7 - 83973087.42\, g^{11} + 0.08907659505\, g -
\end{aligned}
$$

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$$-1685.504320\, g^3 + 1614.901156\, g^5 + 827365423.6\, g^{13} -$$
$$-6852797795.0\, g^{15} + 7721868.666\, g^9 + 34643996560.0\, g^{17}\,.$$

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